

A monotonic conservative finite-difference scheme of second-order accuracy is proposed for the numerical investigation of nonsteady thermoconvective processes.

In the computer modeling of nonsteady thermoconvective processes, the choice of the numerical scheme must satisfy very stringent requirements. Since the solution, as a rule, must be obtained over a long interval of time, it is very important to choose a numerical algorithm that allows the calculation to be carried out with a large time step. Also, to ensure a high accuracy of the solution, the difference scheme must be of a high order of approximation, not only with respect to the space parameters of the grid, but also with respect to the time step.

In order to prevent oscillations of the solution during the calculation process, the difference scheme must be monotonic. Finally, the difference scheme must properly reflect those physical laws which underlie the initial differential equations, i. e., it must be conservative.

In numerical investigations of thermoconvective processes it is usual to start from the Boussinesq hypothesis [1, 2] and to consider a system of equations in the eddy velocity  $\varphi$ , the current function  $\psi$ , and the temperature  $\theta$ ; in plane Cartesian coordinates, this system takes the dimensionless form

$$\frac{\partial \varphi}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\partial \varphi}{\partial x} - u\varphi \right) + \frac{\partial}{\partial y} \left( \frac{\partial \varphi}{\partial y} - v\varphi \right) + Gr \frac{\partial \theta}{\partial x}, \quad (1)$$

$$\varphi = - \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}, \quad u = \frac{\partial \psi}{\partial y}, \quad v = - \frac{\partial \psi}{\partial x}, \quad (2)$$

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x} \left( \frac{1}{Pr} \cdot \frac{\partial \theta}{\partial x} - u\theta \right) + \frac{\partial}{\partial y} \left( \frac{1}{Pr} \cdot \frac{\partial \theta}{\partial y} - v\theta \right). \quad (3)$$

Here  $u$  and  $v$  are the  $x$  and  $y$  components of the velocity vector;  $t$  is the time. For distance the scale  $L$  was taken; for temperature,  $\Delta T$ ; for velocity,  $\nu/L$ ; for time,  $L^2/\nu$ . Both the Prandtl number  $Pr = \nu/\kappa$  and the Grashof number of  $Gr = g\beta L^3 \Delta T/\nu^2$  appear in Eqs. (1)-(3) ( $g$  is the acceleration due to gravity;  $\beta$  is the thermal-expansion coefficient;  $\nu$  is the kinematic viscosity;  $\kappa$  is the thermal conductivity).

The system in Eqs. (1)-(3) has a solution within a closed region  $G$  for definite initial and boundary conditions determined by the specific formulation of the physical problem. At solid impermeable boundaries, for example, there is the adhesion condition

$$\psi = 0, \quad \frac{\partial \psi}{\partial n} = 0. \quad (4)$$

For the temperature, conditions of the first, second, and third kinds may be given, and there also junction conditions. The boundary condition for turbulence is a more complex problem: It cannot be expressed in explicit form and is satisfied approximately in the course of the calculation.

It must be said that those approaches to determining the turbulence at the boundary which, until recently, were universal in computing practice led to significant limitations on the stability and made the application of an economical fractional-step scheme to the solution of Eqs. (1)-(4) largely ineffective. Recently, however, a procedure for calculating the turbulence boundary condition that gives considerably improved accuracy of the

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computational process was proposed [3]. This method significantly increases the efficiency of inexplicit difference schemes of variable-direction type, and allows them to be used to solve problems of convective heat transfer [4].

The distinguishing feature of the proposed numerical scheme is that, being of second-order accuracy, it is monotonic and conservative, and allows calculations to be carried out with  $\tau \sim h$ . Also, the time step  $\tau$  is chosen automatically in the course of the calculation in such a way as to ensure the stability of the computational process.

The difference scheme was obtained by using, in a longitudinal-transverse scheme, the monotonic approximation proposed in [5]. Conservative properties of the scheme were ensured by using the method proposed in [6] for the construction of a steady monotonic conservative scheme of second-order accuracy.

For the model equation

$$\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial x} \left( \frac{1}{\rho} \cdot \frac{\partial \Phi}{\partial x} - u\Phi \right) + \frac{\partial}{\partial y} \left( \frac{1}{\rho} \cdot \frac{\partial \Phi}{\partial y} - v\Phi \right) + f \quad (5)$$

(where  $\rho = \text{const}$ ), the difference scheme has the general form

$$\begin{aligned} \frac{\bar{\Phi}_{i,j} - \Phi_{i,j}^n}{\tau/2} &= \Lambda_x \bar{\Phi} + \Lambda_y \Phi^n + f^{n+1/2}, \\ \frac{\Phi_{i,j}^{n+1} - \bar{\Phi}_{i,j}}{\tau/2} &= \Lambda_x \bar{\Phi} + \Lambda_y \Phi^{n+1} + f^{n+1/2}. \end{aligned} \quad (6)$$

The difference operators  $\Lambda_x$  and  $\Lambda_y$ , approximating the operators  $L_x \Phi = (\partial/\partial x)[(1/\rho)(\partial\Phi/\partial x) - u\Phi]$  and  $L_y \Phi = (\partial/\partial y)[(1/\rho)(\partial\Phi/\partial y) - v\Phi]$ , respectively, are obtained by integrating the latter operators over the elementary volumes  $v_1(x_{i-1/2} \leq x \leq x_{i+1/2}; y_{j-1/2} \leq y \leq y_{j+1/2}; t^n \leq t \leq t^{n+1})$ , and using the Ostrogradskii formula. For example, the operator  $\Lambda_x \Phi^n$  is

$$\begin{aligned} \Lambda_x \Phi^n &= \frac{1}{\rho h} \left( \frac{\Phi_{i+1,j}^n - \Phi_{i,j}^n}{1 + R_{i+1/2}} - \frac{\Phi_{i,j}^n - \Phi_{i-1,j}^n}{1 + R_{i-1/2}} \right) \\ &+ \frac{u_{i-1/2,j}^{n+1/2} + |u_{i-1/2,j}^{n+1/2}|}{2h} \Phi_{i-1,j}^n + \frac{u_{i+1/2,j}^{n+1/2} - |u_{i+1/2,j}^{n+1/2}|}{2h} \Phi_{i,j}^n \\ &- \frac{u_{i+1/2,j}^{n+1/2} + |u_{i+1/2,j}^{n+1/2}|}{2h} \Phi_{i,j}^n - \frac{u_{i-1/2,j}^{n+1/2} - |u_{i-1/2,j}^{n+1/2}|}{2h} \Phi_{i+1,j}^n \end{aligned} \quad (7)$$

where  $R_{i \pm 1/2} = 0.5\rho h |u_{i \pm 1/2,j}^{n+1/2}|$  is the Reynolds difference number. The operator  $\Lambda_y \Phi^n$  is written analogously. The terms  $|u_{i \pm 1/2,j}^{n+1/2}|$ ,  $|v_{i,j \pm 1/2}^{n+1/2}|$  are calculated using the mean values

$$\begin{aligned} u_{i-1/2,j}^{n+1/2} &= \frac{1}{8h} (\psi_{i-1,j+1}^{n+1} + \psi_{i,j+1}^{n+1} - \psi_{i-1,j-1}^{n+1} + \psi_{i,j-1}^{n+1} + \psi_{i-1,j+1}^n + \psi_{i,j+1}^n + \psi_{i-1,j-1}^n - \psi_{i,j-1}^n), \\ v_{i,j-1/2}^{n+1/2} &= \frac{1}{8h} (\psi_{i-1,j-1}^{n+1} + \psi_{i-1,j}^{n+1} - \psi_{i+1,j-1}^{n+1} - \psi_{i,j-1}^{n+1} + \psi_{i-1,j-1}^n + \psi_{i-1,j}^n - \psi_{i+1,j-1}^n - \psi_{i+1,j}^n). \end{aligned} \quad (8)$$

To solve Eq. (2) in each time layer, a variable-direction iterational scheme is used

$$\begin{aligned} \bar{\psi}_{i,j} &= \psi_{i,j}^s + \frac{\sigma}{2} (\bar{\psi}_{xx} + \psi_{yy}^s + \varphi_{i,j}^{n+1}), \\ \psi_{i,j}^{s+1} &= \bar{\psi}_{i,j} + \frac{\sigma}{2} (\bar{\psi}_{xx} + \psi_{yy}^{s+1} + \varphi_{i,j}^{n+1}). \end{aligned} \quad (9)$$

Here  $\sigma$  is the iterational parameter;  $s$  is the number of the iteration;  $\bar{\psi}_{xx}$  and  $\bar{\psi}_{yy}$  are symmetric three-point difference operators approximating the derivatives  $\partial^2\psi/\partial x^2$  and  $\partial^2\psi/\partial y^2$ , respectively.

The calculational process for determining  $\theta_{i,j}^{n+1}$ ,  $\varphi_{i,j}^{n+1}$ ,  $\psi_{i,j}^{n+1}$  from the known values of these parameters at the  $n$ -th time level is constructed as follows. First, from the difference analog of Eq. (3), the value  $\theta_{i,j}^{n+1}$  is determined;  $\psi^n$  is taken as the initial approximation for  $\psi^{n+1,0}$  in Eq. (8). Then Eq. (1) is solved in the region  $G_1 \in G$ , the boundary of which differs from that of  $G$  by one step of the grid. The value of  $\varphi_{1,j}^{n+1}$  on the boundary of  $G_1$  is calculated from the difference equation

$$\varphi_{i,j}^{n+1} = -\psi_{xx}^{n+1} - \psi_{yy}^{n+1}. \quad (10)$$

Inside  $G_1$  the value of  $\varphi_{i,j}^{n+1}$  is determined from the difference analog of Eq. (1).

Next, by means of the iterational process in Eq. (9),  $\psi_{i,j}^{n+1}$  is found inside the main region  $G$ ; at boundary points of the difference grid, the value of  $\psi$  is corrected so as to satisfy the boundary conditions for  $\psi$  and  $\partial\psi/\partial n$ . For example, if the adherence condition  $\psi = \partial\psi/\partial n = 0$  is given at the boundary, then in using a uni-directional three-point approximation of second-order accuracy

$$\left(\frac{\partial\psi}{\partial n}\right)_0 = \frac{-3\psi_0 + 4\psi_1 - \psi_2}{2h} + O(h^2) \quad (11)$$

it is usual to take

$$\psi_0 = 0, \quad \psi_1 = 0.25\psi_2. \quad (12)$$

To refine the values of  $\theta_{i,j}^{n+1}$ ,  $\varphi_{i,j}^{n+1}$ ,  $\psi_{i,j}^{n+1}$  obtained, an external iterational cycle is constructed. Thus, the numerical algorithm includes two iterational cycles: the internal and the external. The internal cycle is terminated by the condition

$$\max_{i,j} \left| \frac{\varphi_{i,j}^{n+1} + \psi_{xx}^{n+1} + \psi_{yy}^{n+1}}{\varphi_{i,j}^{n+1}} \right| \leq \varepsilon_\psi. \quad (13)$$

The external cycle includes the successive solution of all the equations of the system in Eqs. (1)-(3) until the inequality

$$\max_{i,j} \left| \frac{\varphi_{i,j}^{n+1,s_1+1} - \varphi_{i,j}^{n+1,s_1}}{\varphi_{i,j}^{n+1,s_1+1}} \right| \leq \varepsilon_\varphi, \quad (14)$$

where  $s_1$  is the number of the external iteration, is no longer satisfied. Of course, a similar inequality must be satisfied for the function  $\theta$ . However, numerical experiments show that the rate of convergence of  $\theta$  is usually higher than that of  $\varphi$ , and therefore it is reasonable to assume that when Eq. (14) is satisfied the analogous inequality for  $\theta$  will also be satisfied.

Experience of calculations shows that for  $Ra = PrGr \leq 10^4$  it is expedient to set  $\varepsilon_\psi \sim 0.01$  and  $\varepsilon_\varphi \sim 0.05$ . For smaller  $\varepsilon_\psi$  and  $\varepsilon_\varphi$ , the time required for the calculation increases sharply, although the solution is practically unchanged; for larger values, the solution is found to oscillate in the fourth-fifth step. For  $Ra > 10^4$ , it is better to take  $\varepsilon_\psi \sim 0.005$  and  $\varepsilon_\varphi \sim 0.01$ .

The machine time required for the calculation depends significantly on the choice of the iterational parameter  $\sigma$ . In the course of numerical experiments, it was established that the optimal value of  $\sigma$  may be roughly calculated from the formula  $\sigma = h/5$ . With this choice of  $\sigma$ , the required accuracy  $\varepsilon_\psi$  is reached after 5-10 iterations. It would obviously be more expedient to choose iterational parameters optimized according to Jordan [7, 8], but this choice was not made in the present investigations.

It is simple to establish that for  $R_{i \pm 1/2} < 1$  the proposed numerical scheme approximates the initial system of differential equations in Eqs. (1)-(3) with order  $O(\tau^2 + h^2)$ . By summation of the difference equations over all the points of the calculational grid, it may be established that the scheme is conservative.

The systems of difference equations at each of the stages of the calculational process are solved by trial and error. The stability of this method is ensured if

$$\frac{2}{\tau} \geq \max \left( \frac{u_{i-1/2,j}^{n+1/2} - u_{i+1/2,j}^{n+1/2}}{h}, \frac{v_{i,j-1/2}^{n+1/2} - v_{i,j+1/2}^{n+1/2}}{h} \right). \quad (15)$$

The absence in Eq. (15) of restrictions on the spatial parameters of the grid implies that the difference scheme is monotonic. Numerical experiments confirm that the scheme remains monotonic for values of  $Ra$  up to  $10^{10}$  and above.

The stability of the variable-direction scheme in Eq. (6), under the condition of negative determinacy of the operators  $\Lambda_x$  and  $\Lambda_y$ , may be proven using Kellog's lemma [9]. In the case under consideration, negative determinacy of the operators is ensured when Eq. (15) is satisfied. Hence, the scheme is stable for each of

the equations taken individually. However, certain limitations are imposed on  $\tau$  because of the approximation in the boundary conditions for turbulence, which are refined at each time layer. If too large a value of  $\tau$  is taken, the iterations may diverge even when Eq. (15) is satisfied. This is evidently because the region of convergence in the considered iterational process is too small: For large  $\tau$  the initial approximation, usually taken as the value of the function at the previous time level, does not fall within this region, and the process diverges. To prevent the divergence of the iterational process it is necessary to reduce  $\tau$ .

Experience of calculations shows that the convergence of the external iteration is much improved by smoothing out the boundary condition for turbulence, i.e., by writing

$$\varphi_1^{n+1, s_1+1} = (1 - \eta)\varphi_1^{n+1, s_1} + \eta\varphi_1^{n+1, s_1+1}, \quad (16)$$

where  $\eta$  is the smoothing parameter. The optimal value of  $\eta$  depends on the conditions of the problem and is close to 0.85 for  $Ra \leq 10^6$ ; this is in good agreement with the results of [10], where a similar smoothing is used.

At this point, it is perhaps appropriate to point out that, since the boundary condition for the turbulence is unsteady, the matching condition on the intermediate layer must be satisfied, i.e., in this case

$$\bar{\varphi}_1 = \frac{1}{2}(\varphi_1^{n+1} + \varphi_1^n) - \frac{\tau}{4} \Lambda_y (\varphi_1^{n+1} - \varphi_1^n). \quad (17)$$

Violation of Eq. (17) leads to an increase in the machine time for the calculation by approximately 20%. Of course, the same condition must also be satisfied for the temperature if, in the conditions of the problem, it is unsteady on the corresponding portions of the boundary.

Systematic calculations show that, when the indicated conditions are satisfied, the numerical scheme under consideration allows calculations to be carried out with time step  $\tau \sim h$ . However, it is not always possible to carry out calculations with large values of  $\tau$ . For example, in the rapid development of conservative motion, which usually occurs in the initial moments, and also in the reorganization of the convective structure, calculation using large  $\tau$  leads to a sharp increase in the number of external iterations, and a corresponding increase in the calculation time. In such cases, it is better to use a smaller value of  $\tau$  for the calculation.

It is found that the machine time required for the calculation can be very effectively reduced if, in choosing  $\tau$ , the number of external iterations required at the preceding time layer is taken into account. If at the  $n$ -th layer the number of iterations  $s_1^n$  is larger than  $s^{opt}$ , the time step taken at the  $(n+1)$ -th layer is  $\tau^{n+1} = 0.7\tau^n$ . If  $s_1^n \leq s^{opt}$ , the step is increased:  $\tau^{n+1} = 1.2\tau^n$ . If  $s_1^{n+1}$  is larger than  $s^{max}$ , the step is halved; this guarantees the stability of the calculation, since the onset of instability is preceded, as a rule, by a sharp increase in the number of iterations. Experience shows that, if the values  $s^{opt} = 4$ ,  $s^{max} = 8$  are chosen, then on those portions where the solution changes smoothly the program chooses the maximum permissible step  $\tau \sim h$  (the limit is  $\tau < h$ ). The value of  $\tau$  is also regulated by Eq. (15); if Eq. (15) is not satisfied, the step is recalculated according to the formula  $\tau^{n+1} = 0.7\tau^{n+1}$ .

To establish the main features of the presented numerical scheme, systematic calculations were carried out for test problems on the convection in a square region with uniform heating from the side [11] and also with sinusoidal heating from above [12]. In addition, a large number of numerical experiments were performed for nonsteady thermoconvective-wave problems [13].

Monotonic stationary solutions for convection with uniform heating from the side [11] can be obtained for values of  $Ra$  up to  $10^{10}$ . The optimal value of the time step, chosen automatically in the course of the experiment, depends very weakly on the spatial parameters of the grid, which is undoubtedly an advantage of the proposed numerical scheme. For values of  $Ra$  up to  $10^4$ , the mean optimal value of the time step  $\tau$  is found to be approximately equal to  $h$ . With further increase in  $Ra$ , the value of  $\tau$  chosen in the course of the experiment drops markedly.

Investigation of the conservative properties of the numerical scheme, using the example of convection with sinusoidal heating from above [12], shows that for values of  $Ra$  up to  $10^6$  on a grid with  $h = 1/20$ , the calculated values of the thermal fluxes entering and leaving the region differ by no more than 3%. The nonconservative scheme of [14] gives discrepancies of the order of 6-15%.

Numerical investigation of nonsteady thermoconvective-wave problems [13] shows that the scheme responds very well to the nonsteady character of the process. On those portions where the solution changes slowly, the maximum permissible time step is chosen, but where the solution is changing rapidly, the time step is

reduced. By this choice of  $\tau$ , the accuracy of the solution obtained can be maintained more uniformly over the whole of the calculated time interval and, as a result, great economy of the calculations can be achieved.

## NOTATION

$t$ , time;  $\varphi$ , turbulence function;  $\psi$ , current function;  $\theta$ , dimensionless temperature;  $u, v, x$ , and  $y$ , components of velocity;  $L$ , scale of length;  $\Delta T$ , characteristic temperature difference;  $\nu$ , kinematic viscosity;  $\kappa$ , thermal conductivity;  $g$ , acceleration due to gravity;  $\beta$ , thermal-expansion coefficient;  $\tau$ , discretization step for time variable;  $h$ , discretization step for space variables;  $\psi_{xx}^- = (\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j})/h^2$ ;  $n$ , vector normal to surface;  $\sigma$ , iterational parameter;  $\eta$ , smoothing parameter for boundary condition on  $\varphi$ ;  $s$ , number of internal iteration;  $s_1$  number of external iteration;  $\varepsilon_\psi$ , accuracy of internal iteration;  $\varepsilon_\varphi$ , accuracy of external iteration;  $Pr = \nu/\kappa$ , Prandtl number;  $Gr = g\beta L^3 \Delta T/\nu^2$ , Grashof number;  $Ra = PrGr$ , Rayleigh number;  $R_{i\pm 1/2} = 0.5\rho h|u_{i\pm 1,j}|$ , Reynolds difference number.

## LITERATURE CITED

1. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Addison-Wesley (1959).
2. A. V. Lykov and B. M. Berkovskii, *Convection and Thermal Waves* [in Russian], Énergiya, Moscow (1975).
3. V. L. Gryaznov and V. I. Polezhaev, *Dokl. Akad. Nauk SSSR*, 219, No. 2 (1974).
4. V. L. Gryaznov and V. I. Polezhaev, Preprint No. 40, Institute of Applied Mechanics, Academy of Sciences of the USSR, Moscow (1974).
5. A. A. Samarskii, *Zh. Vychisl. Mat. Mat. Fiz.*, 5, No. 3 (1965).
6. B. M. Berkovskii and V. K. Polevikov, *Inzh.-Fiz. Zh.*, 24, No. 5 (1973).
7. B. M. Gerasimov, Preprint No. 13, Institute of Applied Mathematics Academy of Sciences of the USSR, Moscow (1975).
8. A. A. Samarskii, *Introduction to Difference-Scheme Theory* [in Russian], Nauka, Moscow (1971).
9. G. I. Marchuk, *Methods of Computational Mathematics* [in Russian], Nauka, Novosibirsk (1973).
10. C. E. Pearson, *J. Fluid Mech.*, 21, No. 4 (1965).
11. E. L. Tarunin, in: *Hydrodynamics* [in Russian], Vol. 1, No. 184, Perm' (1968).
12. B. M. Berkovskii and E. F. Nogotov, *Dokl. Akad. Nauk SSSR*, 209, No. 1 (1973).
13. B. M. Berkovskii and A. K. Sinitsyn, *Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza*, No. 1 (1975).
14. A. V. Lykov, B. M. Berkovskii, and E. F. Nogotov, in: *Heat and Mass Transfer* [in Russian], Vol. 8, Minsk (1972).

## HEAT-CONDUCTION PROBLEM FOR A MULTIPLY CONNECTED BODY

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A method for solving a heat-conduction problem for multiply connected domains is proposed based on consecutive solution of problems for doubly connected domains. To provide an example the heat-conduction problem is solved for a circle with two circular holes.

In applied mathematics the evaluation of temperature fields in multiply connected domains is a very difficult problem. As mentioned in [1] there is no universal analytic method which would ensure a solution to a heat-conduction problem. The possibilities of numerical methods are wide; their implementation, however, meets with difficulties, and to overcome them one must, as a rule, analyze each problem separately. An approach which would reduce the solution of a heat-conduction problem for a multiply connected domain to the solving of several problems of the same kind would, therefore, be welcome. The method proposed

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